Applications of generalized coherent states in Bose-Hubbard model

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Overview

1. **Bose-Hubbard model**

2. **Generalized coherent state**

3. **Time-dependent variational principle**

Bose-Hubbard model

$\hat{H} = -J\sum_{\langle i,j\rangle}(a_i^\dagger a_j + \text{h.c.}) + \frac{U}{2}\sum_{i=1}^{M} a_i^{\dagger 2} a_i^2,$

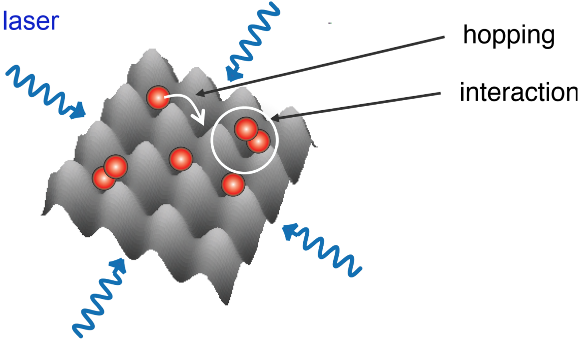


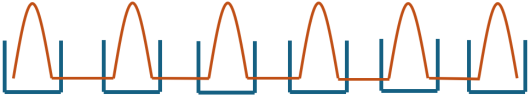
Figure is from Peter Zoller's talk

Bose-Hubbard model

Phase transition of ground state depends on the ratio: $\frac{U}{J}$

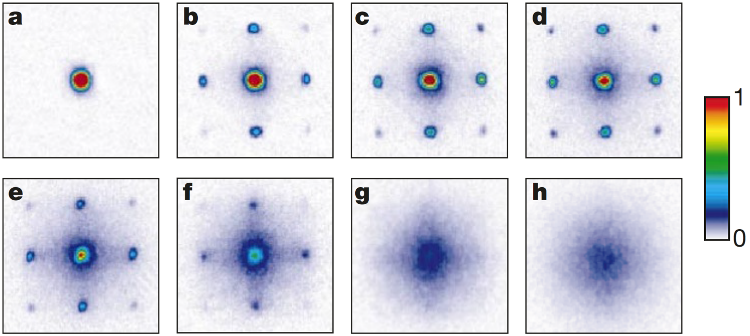
for $U/J \to \infty$, $|\mathrm{GS}\rangle = \prod_{i=1}^{M} a_i^\dagger |0\rangle = |11\cdots 1\rangle$ (M is equal to number of particles S)

for $U/J \to 0$, $|\mathrm{GS}\rangle = \frac{1}{\sqrt{S!}} \left(\sum_{i=1}^{M} \frac{1}{\sqrt{M}} a_i^\dagger\right)^S |00\cdots 0\rangle = \frac{1}{\sqrt{S!}} \left(b_{p=0}^\dagger\right)^S |0\rangle$



Bose-Hubbard model

Phase transition from superfluid to Mott insulator:



M. Greiner, et al., Nature 415, 39–44 (2002)

Generalized coherent state (GCS)

$|S, \boldsymbol{\xi}\rangle = \frac{1}{\sqrt{S!}} \left(\sum_{i=1}^{M} \xi_i \hat{a}_i^\dagger \right)^S |\text{vac}\rangle$, S: number of particles, M: number of modes/sites

$\boldsymbol{\xi} = \{\xi_1, \xi_2, \cdots, \xi_M\}$, normalization condition: $\sum_{i=1}^{M} |\xi_i|^2 = 1$

$\hat{b}^\dagger = \sum_{i=1}^{M} \xi_i \hat{a}_i^\dagger$, $|S, \boldsymbol{\xi}\rangle = \frac{1}{\sqrt{S!}} \left(\hat{b}^\dagger\right)^S |\text{vac}\rangle$: condensate

Example: $|2, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle = \frac{1}{2}|20\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{2}|02\rangle$

P. Buonsante and V. Penna, J. Phys. A: Math. Theor. 41, 175301 (2008)

Relationship between two ground states

$|11\cdots 1\rangle = \sum_{k=1}^{N} A_k |S, \xi_k\rangle$, $N = 2^{S-1}$

Simple equation: $x_1 x_2 = \frac{1}{4}(x_1 + x_2)^2 - \frac{1}{4}(x_1 - x_2)^2$

general case: $x_1 x_2 \cdots x_M \rightarrow$ Kan's formula

Sinle parameterized state can not describe the phase transtion

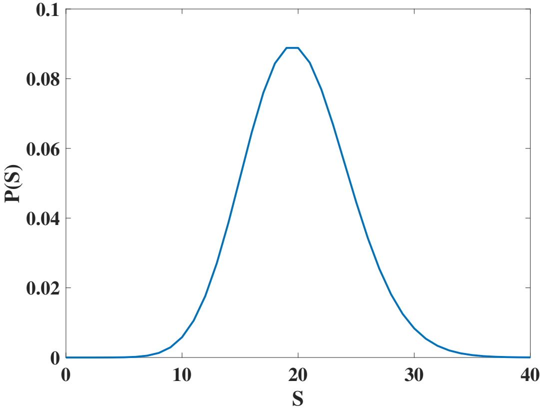
R. Kan, J. Multivar. Anal. 99, 542 (2008).

Y. Q, J. Huh and F. Grossmann, SciPost Phys. 15, 007 (2023)

GCS / Glauber coherent state

$$|\vec{\alpha}\rangle = \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{M}|\alpha_i|^2}\mathrm{e}^{\sum_{i=1}^{M}\alpha_i\hat{a}_i^\dagger}|0,0,\cdots,0\rangle$$
$$= \sum_{S=0}^{\infty}\sqrt{P(S)}|S,\vec{\xi}\rangle$$

$$\tilde{N} = \sum_{i=1}^{M}|\alpha_i|^2, \quad P(S) = \mathrm{e}^{-\tilde{N}}\frac{\tilde{N}^S}{S!}$$



GCS / Glauber coherent state

Multi-mode Glauber coherent state	generalized coherent state
$\|\boldsymbol{\alpha}\rangle = \prod_{i=1}^{M} e^{-\frac{\|\alpha_i\|^2}{2}} e^{\alpha_i a \hat{a}_i^\dagger} \|\mathrm{vac}\rangle = \otimes_{i=1}^{M} \|\alpha_i\rangle$	$\|S,\boldsymbol{\xi}\rangle = \frac{1}{\sqrt{S!}} \left(\sum_{i=1}^{M} \xi_i \hat{a}_i^\dagger\right)^S \|\mathrm{vac}\rangle$
$a_i\|\boldsymbol{\alpha}\rangle = \alpha_i\|\boldsymbol{\alpha}\rangle$	$\hat{a}_i\|S,\boldsymbol{\xi}\rangle = \sqrt{S}\xi_i\|S-1,\boldsymbol{\xi}\rangle$
$\langle\boldsymbol{\alpha}\|\hat{a}_i^\dagger a_j\|\boldsymbol{\alpha}\rangle = \alpha_i^* \alpha_j$	$\langle S,\boldsymbol{\xi}\|\hat{a}_i^\dagger \hat{a}_j\|S,\boldsymbol{\xi}\rangle = S\xi_i^*\xi_j$
$\langle\boldsymbol{\alpha}\|\boldsymbol{\beta}\rangle = \exp\left[\sum_{i=1}^{M} \alpha_i^*\beta_i - \frac{1}{2}(\|\alpha_i\|^2 + \|\beta_i\|^2)\right]$	$\langle S',\boldsymbol{\xi}\|S,\boldsymbol{\eta}\rangle = \left(\sum_{i=1}^{M} \xi_i^*\eta_i\right)^S \delta_{S,S'}$
overcomplete in whole Hilbert space	overcomplete in S-particle subspace

Time-dependent variational principle

Initial state: $|\psi(t=0)\rangle = |S,\boldsymbol{\xi}\rangle$

ansatz: $|\psi(t)\rangle = \sum_{k=1}^{N} A_k(t)|S,\boldsymbol{\xi}_k(t)\rangle$, N: multiplicity.

- GCS serve as overcomplete bases
- $N=1$: $\mathrm{i}\dot{\xi}_i = -J(\xi_{i+1}+\xi_{i-1}) + U(S-1)|\xi_i|^2\xi_i$
- $N>1$ is beyond mean-field result if $\langle S,\boldsymbol{\xi}_k|S,\boldsymbol{\xi}_{k'}\rangle \neq 0$

Y. Q and F. Grossmann, Phys.Rev.A 103, 042209 (2021)

Time-dependent variational principle

Parameterized ansatz: $|\psi(\boldsymbol{x})\rangle = |\psi(x_1, x_2, \cdots, x_M)\rangle$, $\boldsymbol{x} = \{\boldsymbol{A}, \boldsymbol{\xi}\}$

Schrodinger equation: $\partial_t|\psi(\boldsymbol{x})\rangle \approx -i\hat{H}|\psi(x)\rangle$

Left side: $\partial_t|\psi(\boldsymbol{x})\rangle = \dot{x}^\nu|V_\nu\rangle$, $|V_\nu\rangle = \partial_{x_\nu}|\psi\rangle$

Right side: $-i\hat{H}|\psi(x)\rangle$

Best approximation: $\langle V_\mu|V_\nu\rangle\dot{x}^\nu = -i\langle V_\mu|\hat{H}|\psi(x)\rangle$

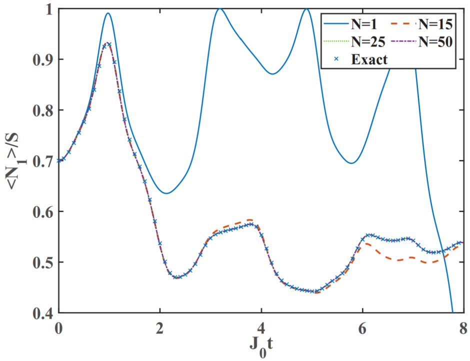
L. Hackl, et al., SciPost Phys. 9, 048 (2020).

Nonequilibrium dynamics

Two-site mode:

$$H = -J(t)(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{U}{2}(\hat{a}_1^{\dagger 2}\hat{a}_1^2 + \hat{a}_2^{\dagger 2}\hat{a}_2^2), \; J(t) = J_0 + J_1 \cos(\omega t)$$

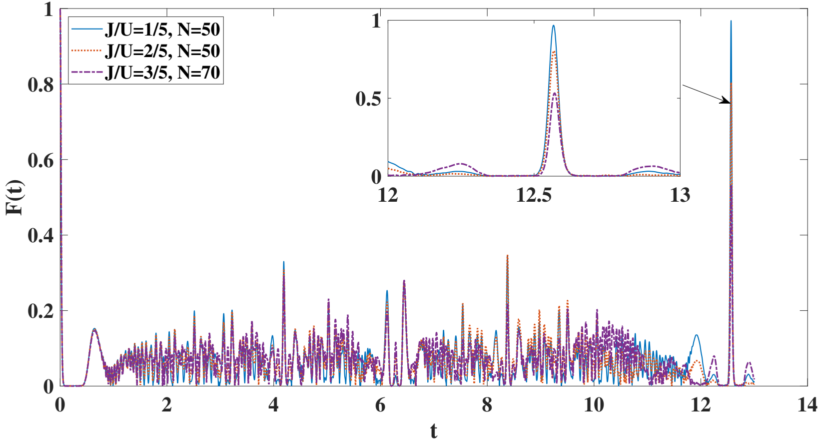
initial state: $|S = 50, \xi_1 = -\sqrt{0.7}, \xi_2 = \sqrt{0.3}\rangle$



Nonequilibrium dynamics

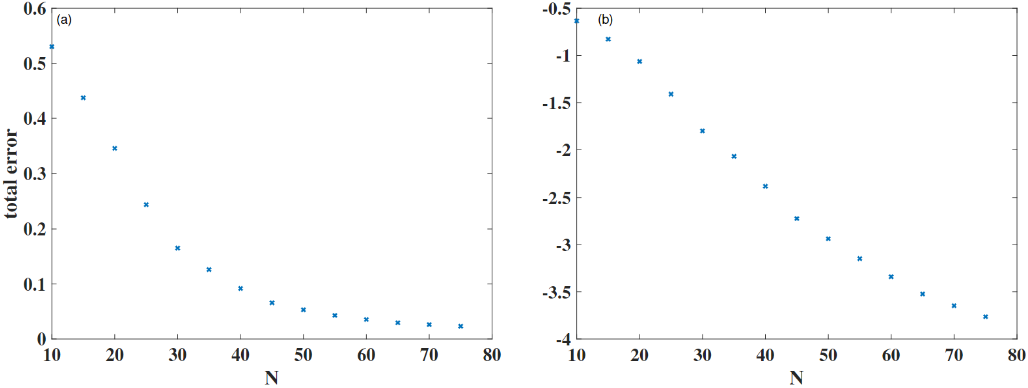
Considering initial state: $|\vec{\alpha}\rangle = |\sqrt{20}, 0, \sqrt{20}, 0\rangle$ with $\tilde{N} = 40,$ if $J = 0,$ $T = \frac{2\pi}{U}$

$F(t) = |\langle\vec{\alpha}|e^{-i\hat{H}t}|\vec{\alpha}\rangle|,$ **S. Tomsovic, et al, Phys.Rev.A, 2018, 97(6)**



Nonequilibrium dynamics

Accumulated error: $E = \int_0^t \mathrm{d}\tau |F_{\mathrm{GCS}}(\tau) - F_{\mathrm{exact}}(\tau)|$



Outlook

Linear combination of GCS for

1. Extended BH model/time-dependent Hamiltonian/...

2. Scrambling of quantum information (OTOC)

3. Fermi systems ......